International Journal of Theoretical Physics, Vol. 45, No. 7, July 2006 (© 2006)
DOI: 10.1007/s10773-006-9148-6

Exchangeability and Conditionally Identical Common Cause Systems

Gábor Hofer-Szabó

Received November 4, 2005; accepted March 13, 2006
Published Online: June 9, 2006

A pair (A, B) of events in a classical probability measure space (Ω, p) is called exchangeable iff $p(A\overline{B}) = p(\overline{A}B)$. Conditionally identical common cause system of size n for the correlation is an n-partition of Ω such that (i) any member of the partition screens the correlation off and (ii) for any member $\{C_i\}_{i\in I}$ of the partition $p(A|C_i) = p(B|C_i)$. The common cause system is called proper if $p(A|C_i) \neq (A|C_j)$ for some $i \neq j$. In the paper it is shown that exchangeable correlations be explained by proper conditionally identical common cause systems in the following sense. (i) Given a proper conditionally identical common cause system of size n for the two events A and B in Ω, then the pair (A, B) will be an exchangeable (positively) correlating pair. (ii) Given any exchangeable (positively) correlating pair of events in Ω and given any finite number $n > 2$, then the probability space can be embedded into a larger probability space in such a way that the larger space contains a proper conditionally identical common cause system of size n for the correlation.

KEY WORDS: Reichenbachian common cause; exchangeability; correlation.

1. INTRODUCTION

Let S be a pair of spin-$\frac{1}{2}$ particles prepared in the singlet state $|\Psi_s\rangle$ in the usual EPR experimental setup. Let $p(A_1)$ denote the probability that the spin measurement on particle 1 in direction a yields the result $+1$ and let $p(\overline{A}_1)$ denote the probability that the measurement on particle 1 in direction **a** yields the result -1. Let $p(B_2)$ and $p(\overline{B}_2)$ be defined in a similar way for particle 2 in direction **b**. For the joint probability $p(A_1\overline{B}_2)$ quantum mechanics predicts

$$p(A_1\overline{B}_2) = \frac{1}{2}\sin^2\frac{1}{2}(\pi - \Theta_{\mathbf{a},\mathbf{b}})$$

[1] Department of Philosophy and History of Science, Budapest University of Technology and Economics, Hungary; e-mail: gszabo@hps.elte.hu.

where $\Theta_{\mathbf{a,b}}$ denotes the angle between the two measuring directions. Since $p(A_1)$ and $p(\overline{B}_2)$ are both $\frac{1}{2}$, the correlation

$$\mathrm{Corr}_p(A_1, \overline{B}_2) = p(A_1\overline{B}_2) = p(A_1)p(\overline{B}_1)$$

will be positive if $\Theta_{\mathbf{a,b}} \in [0, \frac{\pi}{2})$, negative if $\Theta_{\mathbf{a,b}} \in (\frac{\pi}{2}, \pi]$ and zero (that is A_1 and $\overline{B}_2$ are independent) if $\Theta_{\mathbf{a,b}} = \frac{\pi}{2}$.

Since spin is a two-valued observable and the experimental setup has axial symmetry (that is $p(A_1B_2)$ depends solely on the the angle $\Theta_{\mathbf{a,b}}$ between the two measuring directions) we get the same result for the joint probability $p(\overline{A}_1, B_2)$ that is

$$p(\overline{A}_1 B_2) = \frac{1}{2}\sin^2\frac{1}{2}(\pi - \Theta_{\mathbf{a,b}})$$

Since $\mathrm{Corr}_p(\overline{A}_1, B_2) = \mathrm{Corr}_p(A_1, \overline{B}_2)$, the sign of $\mathrm{Corr}_p(\overline{A}_1, B_2)$ will depend on the value of $\Theta_{\mathbf{a,b}}$ just as before.

Here in the paper we disregard the special quantum mechanical details of the EPR experiment, rather we concentrate on two general features of the spin measurement. The one is the symmetry property

$$p(A_1\overline{B}_2) = p(\overline{A}_1 B_2)$$

which we later call exchangeability, the other feature is simply the sign of the correlation.

In Section 2 we define the notions of Reichenbachian common cause and Reichenbachian common cause system, respectively as the most promising candidates for a common-cause-type explanation of such correlating events which do not influence each other causally. In Section 3 exchangeability will be defined and some motivations will be given from a subjectivist account of probability. Then the Hypothesis of Conditionally Identical Probabilities will be put forward as a causal explanation of exchangeable correlating pairs of events. In Section 4 we state and prove our main propositions concerning the necessary and sufficient condition of a conditionally-identical-common-cause-type explanation of an exchangeable correlation. In the Conclusions the metaphysical status of the Hypothesis of Conditionally Identical Probabilities will be discussed.

2. REICHENBACHIAN COMMON CAUSES AND COMMON CAUSE SYSTEMS

Let (Ω, p) be a classical probability measure space and let $A, B \in \Omega$ be two positively correlating events, i.e.

$$\mathrm{p(AB)} > \mathrm{p(A)p(B)} \tag{1}$$

and let the quantity

$$\mathrm{Corr}_p(A, B) \equiv p(AB) - p(A)p(B)$$

be called the *correlation* of A and B in (Ω, p).

In (1956) Reichenbach defines the common cause of the correlation as follows:

Definition 2.1. *An event* C *in* Ω *is said to be the* (Reichenbachian) common cause *of the correlation between* A *and* B *if the events* A, B *and* C *satisfy the following relations:*

$$p(AB|C) = p(A|C)p(B|C) \tag{2}$$

$$p(AB|\overline{C}) = p(A|\overline{C})p(B|\overline{C}) \tag{3}$$

$$p(A|C) > p(A|\overline{C}) \tag{4}$$

$$p(B|C) > p(B|\overline{C}) \tag{5}$$

where $p(X|Y) = p(XY)/p(Y)$ denotes the conditional probability of X on condition Y, $\overline{C}$ denotes the complement of C and it is assumed that none of the probabilities is equal to zero. Equations (2)-(3) are called "screening-off" properties since conditioning on C and $\overline{C}$, respectively screens off the correlation between A and B. (4)-(5) express the "positive statistical relevance" of the cause C on the two effects A and B, respectively.
What is the situation if the correlation is due not only to a single causal factor but to a system of different causal effects? In other words, how can the notion of the Reichenbachian common cause be generalized for situations when more causes are present? The idea is the following: explaining a correlation by a system of common causes would mean that one can partition the statistical ensemble into more than two subensembles in such a manner that (i) the correlation disappears in each of the subensembles, (ii) any pair of such subensembles behaves like the two subensembles determined by the pair $\{C, \overline{C}\}$ in the Definition 1 of the common cause. For other motivations of the definition we refer the reader to (Hofer-Szabó and Rédei, 2004, 2006). A mathematically explicit formulation of this idea is spelled out in the next definition.

Definition 2.2. *Let* (Ω, p) *be a probability space and* A, B *two events in* Ω. *The partition* $\{C_i\}_{i\in I}$ *of* Ω *is said to be a* Reichenbachian common cause system *for the pair* (A, B) *if for all* $i, j \in I (i \neq j)$ *the following two conditions are satisfied*

$$p(AB|C_i) = p(A|C_i)p(B|C_i) \tag{6}$$

$$(p(A|C_i) - p(A|C_j))(p(B|C_i) - p(B|C_j)) > 0 \tag{7}$$

The above definition is a natural generalization of Reichenbach's original definition of common cause to the case when more than one single factor contributes to the correlation. The cardinality of the index set I (i.e. the number of events in the partition) is called the *size* of the Reichenbachian common cause system. It is straightforward to see that a Reichanbachian common cause $\{C, \overline{C}\}$ is a Reichenbachian common cause system of size 2.

In what follows we define the notion of exchangeability and use the concept of the Reichenbachian common cause system for the common-causal explanation of exchangeable correlations.

3. EXCHANGEABILITY AND IDENTITY OF CONDITIONAL PROBABILITIES

Definition 3.3. *Let (Ω, p) be a classical probability measure space. A pair (A, B) of events in Ω is said to be* exchangeable *if*

$$\mathrm{p}(\mathrm{A}\overline{B}) = \mathrm{p}(\overline{A}B) \tag{8}$$

Exchangeability was introduced by de Finetti (1938) and came to be the central notion in the subjectivist account of probability. In their liked coin-tossing examples exchangeability means that the probability of getting k heads in a sequence of n tosses does not depend on which trial heads occured on but it depends solely on the k number of heads in the sequence. Generally, a set of random variables is said to be exchangeable if their joint disribution is invariant under permutations of the sequence of the variables (Jeffrey, 2004). Exchangeability in subjectivist theories of probability replaces the notion of independence. Subjectivists prefer exchangeability to independence since exchangeability yields a formal basis to subjectivist account of induction. Laplace's Rule and other rules of succession can be backed by exchangeability in the subjectivist framework.

Here we do not commit ourselves to the subjectivist account of probability but rather we use the notion of exchangeability in a formal way. In Definition 3 we defined exchangeability on the level of the events and measures not on the level of the distributions of random variables as usual. Exchangeability expresses a special permutation symmetry between two events: the probability that the one event occurs and the other event does not occur, does not depend on which event occurs and which does not. In order to spell out this symmetry consider the partition $\{AB, A\overline{B}, \overline{A}B, \overline{AB}\}$ generalized by the events A and B in Ω and consider the following vector function on $\Omega \times \Omega$:

$\mathrm{f}_p : \Omega \times \Omega \longrightarrow \mathbb{R}^4; (A, B) \longmapsto \{p(AB), p(A\overline{B}), p(\overline{A}B), p(\overline{AB})\}$ Now exchangeability is the invariance of f_p under the S_2 permutation $A \leftrightarrow B$, or in other words, a pair (A, B) of events in (Ω, p) is exchangeable if $f_p(A, B) = f_p(B, A)$.

Physical motivations for exchangeability must be clear from the spin experiment mentioned above. For two-particles system in singlet state with axial symmetry the outcomes of the measurement of a two-valued observable are exchangeable.

Next we connect exchangeability and correlation in a straightforward way:

Definition 3.4. *A pair of events* A *and* B *in* (Ω, p) *is said to be an* exchangeable (positively) correlating pair *if the followings hold:*

$$\mathrm{p(AB)} \ > \ \mathrm{p(A)p(B)} \tag{9}$$

$$\mathrm{p(A}\overline{B}) = p(\overline{A}B) \tag{10}$$

It is easy to see that if (A, B) *is an exchangeable correlating pair then* $p(A) = p(B)$.

Now we turn to the question of how to explain exchangeable correlations if the correlating events do not interact causally. Since the most promising candidate for common-cause-type explanation of correlations is the Reichenbachian common cause system, in what follows, we impose some extra requirements on the Reichenbachian common cause system such that the common cause system accounts for the exchangeable character of the correlation.

Suppose A and B are two exchangeable correlating events and suppose that $\{C_i\}_{i\in I}$ is a Reichenbachian common cause system that explains the correlation in the sense that A, B and $\{C_i\}_{i\in I}$ satisfy (6)-(7). Since exchangeability is an invariance under the permutation $A \leftrightarrow B$ it is natural to *assume* that the exchangeability of the correlating pair derives from the same type of invariance of the underlying causal structure, namely the invariance of the Reichenbachian common cause system under the transformation $A \leftrightarrow B$. If the causal source of the correlation is invariant under the transformation $A \leftrightarrow B$, then the correlation will inherit this invariance and so the pair (A, B) will be exchangeable.

How can the invariance of the causal source of the correlation under the transformation $A \leftrightarrow B$ be defined? Since in this statistical framework the strenght of the causal efficiency of the common cause system on the events A and B is measured by the conditional probabilities $p(A|C_i)$ and $p(B|C_i)$, the invariance of the causal effect under the transformation $A \leftrightarrow B$ simply means that the conditional probabilities $p(A|C_i)$ and $p(B|C_i)$ are identical. This notion of *conditional identity* is spelled out in the following definition:

Definition 3.5. *Let* A, B *be two events in* (Ω, p) *and let* $\{C_i\}_{i\in I}$ *be a partition of* Ω. *Then* $\{C_i\}_{i\in I}$ *is said to be a* conditionally identical partition *with regard to* A *and* B *if for all* $i \in I$

$$p(A|C_i) = p(B|C_i)\ . \tag{11}$$

Conditional identitical partitions are invariant under the transformation $A \leftrightarrow B$. Using Definition 5 now we can define *conditionally identical common cause system* as follows:

Definition 3.6. *Let* (A, B) *be a correlating pair of events in a classical probability measure space* (Ω, p). *Then a partition* $\{C_i\}_{i \in I}$ *of* Ω *is said to be a* conditionally identical common cause system *of the correlation* (A, B) *if for all* $\{C_i\}_{i \in I}$ *the following equations hold:*

$$p(AB|C_i) = p(A|C_i)p(B|C_i) \tag{12}$$

$$p(A|C_i) = p(B|C_i) \tag{13}$$

We call a conditionally identical common cause system *proper* iff $p(A|C_i) \neq p(A|C_j)$ for some $i \neq j$. The cardinality of the index set I is called as before the *size* of the common cause system.

Definition 6 of the conditionally identical common cause system has to meet two demands: first, it has to explain the correlation between the events A and B; second, it has to account for the exchangeability of the pair. Both requirements are fullfilled. Since (13) implies (7), conditionally identical common cause systems are Reichenbachian common cause systems, and so they are appropriate tools for explaining correlations. On the other hand, (13) expresses conditional identity which—according to our assumption—is responsible for the exchangeability of the pair (A, B).

Having defined the notion of conditionally identical common cause system now we can make explicite our assumption mentioned above. This is spelled out in the following hypothesis:

Hypothesis of Conditionally Identical Probabilities: Exchangeability of a (causally not interacting) correlating pair derives from a conditionally identical common cause system.

In the following section we make the Hypothesis of Conditionally Identical Probabilities more precise and decide on its truth value. In the Conclusions we return to the metaphysical interpretation of the Hypothesis.

4. CONDITIONALLY IDENTICAL COMMON CAUSE SYSTEMS AND POSITIVE CORRELATION

The precise content of the Hypothesis of Conditionally Identical Probabilities can be expressed as follows. Let be (A, B) an exchangeable correlating pair in a classical probability measure space (Ω, p). Then a *necessary and sufficient* condition that there exists a proper conditionally identical common cause system $\{C_i\}_{i\in I}$ for the pair (A, B) is that there exists a positive correlation between A and B.

In order to prove necessity we have to show that the existence of a proper factorizing partition with identical conditional probabilities for an exchangeable correlating pair (A, B) implies positive correlation between A and B. In order to prove sufficiency we have to show that if (A, B) is an exchangeable positively correlating pair in (Ω, p), then there exists an extension (Ω', p') of (Ω, p) such that (Ω', p') contains a proper conditionally identical common cause system of size n for the correlation of A and B.

These tasks will be accomplished in the following two theorems:

Theorem 4.1. *Let* A *and* B *be events in a classical probability measure space* (Ω, p) *and let* $\{C_i\}_{i\in I}$ *be a partition of* Ω. *Furthermore, let* (AB) *be a conditionally identical common cause system of the pair* (A, B). *Then* (A, B) *is an exchangeable positively correlating pair.*

Proof: First we prove positive correlation, then exchangeability. Since $\{C_i\}_{i\in I}$ is a conditionally identical common cause system of the pair (A, B) in (Ω, p), the events A, B and $\{C_i\}_{i\in I}$ satisfy (12)–(13). Using conditional decomposition of the correlation

$$\mathrm{Corr}(A, B) \equiv p(AB) - p(A)p(B) = \sum_{i<j} p(C_i)p(C_j)[p(A|C_i) - p(A|C_j)][p(B|C_i) - p(B|C_j)]$$

and Equation (13) we get

$$\mathrm{Corr}(A, B) = \sum_{i<j} p(C_i)p(C_j)[p(A|C_i) - p(A|C_j)]^2$$

which is positive for proper factorizing partitions regardless of the values of $p(C_i)$. So the correlation between A and B will be positive.

Now we turn to the proof of exchangeability. Using (12) and the conditional decomposition of $p(A\overline{B})$ we get

$$p(A\overline{B}) = \sum_i p(C_i)p(A|C_i)p(\overline{B}|C_i)$$

Using (13) and substituting $p(A|C_i)$ by $p(B|C_i)$ and $p(\overline{B}|C_i)$ by $p(\overline{A}|C_i)$ we get

$$\sum_i p(C_i)p(B|C_i)p(\overline{A}|C_i)$$

which is just $p(\overline{A}B)$. Thus a conditionally identical common cause system for a pair of events implies exchangeability of the pair.

Theorem 4.2. *Let* A *and* B *exchangeable events in a classical probability measure space* (Ω, p) *and let there be a positive correlation between* A *and* B. *Then for any given* $n > 2$ *real number the probability measure space* (Ω, p) *can be extented such that the extension* (Ω', p') *contains a proper conditionally identical common cause system of size* n *for the correlation.*

Proof: The proof consists of two steps. In Step 1 we show first that if $\{C_i\}_{i=1}^n$ is a conditionally identical common cause system of size n in a probability space (Ω, p), then the numbers $\{p(A|C_i)\}_{i=1}^n$, $\{p(B|C_i)\}_{i=1}^n$ and $\{p(C_i)\}_{i=1}^n$ satisfy certain equations and inequalities. Since the Reichenbachian common cause system $\{C_i\}_{i\in I}$ is conditionally identical that is $p(A|C_i) = p(B|C_i)$ for all $i = 1 \ldots n$ we substitute $p(B|C_i)$ by $p(A|C_i)$ in the equations and inequalities and omit those that become redundant by the substitution. It is then shown that (if the correlation is not strict) then there exist $2n$ non-negative real numbers $\{a_i\}_{i=1}^n$ and $\{c_i\}_{i=1}^n$ that satisfy those equations. We call the numbers $\{a_i\}_{i=1}^n$, $\{c_i\}_{i=1}^n$ *admissible numbers* for the correlation $\mathrm{Corr}_p(A, B)$. Finally we identify the admissible numbers with the numbers $\{p(A|C_i) = p(B|C_i)\}_{i=1}^n$ and $\{p(A|C_i)\}_{i=1}^n$ via

$$a_i = p(A|C_i) = p(B|C_i) \quad i = 1, \ldots, n \tag{14}$$

$$c_i = p(C_i) \qquad i = 1, \ldots n \tag{15}$$

In Step 2 we show that, given a non-strict correlation in any probability space and any n, for any given set of admissible numbers for $\mathrm{Corr}_p(A, B) > 0$, there exists an extension (Ω', p') of (Ω, p) such that (Ω', p') contains a proper conditionally identical common cause system of size n in such a way that the admissible numbers are identified with the corresponding probabilities via (14)–(15). For a more general result we refer the reader to (Hofer-Szabó and Rédei, 2006).

Step 1 Let $\{C_i\}_{i=1}^n (n \geq 2)$ be a conditionally identical common cause system in (Ω, p) of the correlation $\mathrm{Corr}_p(A, B) > 0$.

Using the theorem of total probability and (12)–(13), the probabilities $p(A)$, $p(B)$ and $p(AB)$ can be written as

$$p(A) = p(B) = \sum_{i=1}^{n} p(A|C_i)p(C_i) \tag{16}$$

$$p(AB) = \sum_{i=1}^{n} p(A|C_i)^2 p(C_i) \tag{17}$$

Obviously, one also has

$$1 = \sum_i p(C_i) \tag{18}$$

$$0 \leq p(A|C_i) \leq 1 \quad (i = 1, \dots n) \tag{19}$$

$$0 < p(C_i) < 1 \quad (i = 1, \dots n) \tag{20}$$

Hence the assumption of a conditionally identical common cause system $\{C_i\}_{i=1}^n$ of the correlation between A and B implies that there exist $2n$ real numbers $\{a_i\}_{i=1}^n$, $\{c_i\}_{i=1}^n$ such that with the identifications (14)–(15) the requirements (16)–(20) hold, i.e. there exist real numbers $\{a_i\}_{i=1}^n$, $\{c_i\}_{i=1}^n$ for which we have

$$p(A) = p(B) = \sum_{i=1}^n a_i c_i \tag{21}$$

$$p(AB) = \sum_{i=1}^n a_i^2 c_i \tag{22}$$

$$1 = \sum_i c_i \tag{23}$$

$$0 \leq a_i \leq 1 \quad (i = 1, \dots n) \tag{24}$$

$$0 < c_i < 1 \quad (i = 1, \dots n) \tag{25}$$

Given a correlation $\mathrm{Corr}_p(A, B) > 0$ in a probability space (Ω, p), the set

$$\{a_i\}_{i=1}^n \qquad \{c_i\}_{i=1}^n \tag{26}$$

of $2n$ real numbers is called an *admissible* set for $\mathrm{Corr}_p(A, B) > 0$ if (21)–(25) hold. We now show that given any non-strict correlation $\mathrm{Corr}_p(A, B) > 0$ in a probability space (Ω, p), and given any $n \geq 2$, there exist a set of admissible numbers for the correlation.

Given a set $\{a_i\}_{i=1}^n$, $\{c_i\}_{i=1}^n$ of real numbers that satisfy (24)–(25), the three Equations (21)–(23) restrict the number of independent numbers to $(2n - 3)$. Consider the following $(2n - 3)$ numbers

$$\{a_i\}_{i=1}^{n-1}, \quad \{c_i\}_{i=1}^{n-2} \tag{27}$$

as parameters. A routine calculations shows that, using (21)–(23) the numbers a_n, c_n and c_{n-1} can be expressed with the help of parameters (27) as follows:

$$c_{n-1} = \frac{p(AB) - p(A)^2 + S_k^{n-2}(A) - D_k^{n-2}(A) + S^{n-2}(A)}{[p(A) - a_{n-1})]^2 + p(AB) - p(A) - p(A)^2 + A_{n-1,k}^{n-2}} \tag{28}$$

$$c_n = 1 - \sum_{i=1}^{n-1} c_i \tag{29}$$

$$a_n = \frac{p(A) - T_k^{n-2}(A) - a_{n-1}}{1 - S^{n-2} - c_{n-1}} + a_{n-1} \tag{30}$$

where

$$S_{j,k}^{n-2}(A) = \frac{1}{2} \sum_{j,k=1}^{n-2} c_j c_k [a_j - a_k]^2$$

$$S_k^{n-2}(A) = \sum_{k=1}^{n-2} c_k [p(A) - a_k]^2$$

$$S^{n-2}(A, B) = \sum_{k=1}^{n-2} c_k [p(AB) - p(A)^2)$$

$$S_{n-1,k}^{n-2}(A) = \sum_{k=1}^{n-2} c_k [a_{n-1} - a_k]^2$$

$$S^{n-2} = \sum_{k=1}^{n-2} c_k$$

$$T_{n-1,k}^{n-2}(A) = \sum_{k=1}^{n-2} c_k [a_{n-1} - a_k]$$

It follows that if one can show that one can choose the parameters (27) in such a way that c_{n-1}, c_n, a_n given by (28)–(30) satisfy (24)–(25), then one has shown that $2n$ admissible numbers exist. We show this by induction on n: it is proved first that non-zero admissible numbers for $n = 2$ exist, then it is assumed that for $n \geq 2$ $2n$ non-zero admissible numbers exist and then, using a simple continuity argument, it is shown, that $2(n + 1)$ admissible numbers exist.

Consider the case $n = 2$. Let us first define the following domain $\mathcal{D} \subseteq R$:

$$\mathcal{D} = (0, p(B|\overline{A})) \cup (p(B|A), 1) \tag{31}$$

Since the correlation is not strict, $\mathcal{D}$ is not empty. Since $n = 2$, in this case the parameter is a_i, and all the S's and T's are zero, hence (28)–(30) reduce to the following

$$c_1 = \frac{p(AB) - p(A)^2}{[p(A) - a_1)]^2 + p(AB) - p(A)^2} \tag{32}$$

$$c_2 = 1 - c_1 \tag{33}$$

$$a_2 = \frac{p(A) - a_1}{1 - c_1} + a_1 \tag{34}$$

It is straightforward to verify that if a_1 is in $\mathcal{D}$, then (24)–(25) are also satisfied. Thus the parameter $0 < a_1$ determines an admissible set

$$\{a_1, a_2, c_1, c_2\} \tag{35}$$

of *non-zero* numbers for the correlation $\mathrm{Corr}_p(A, B) > 0$ in the case of $n = 2$.

Let now $n > 2$ be arbitrary and let us assume (inductive hypothesis) that we have a set

$$\{a_i\}_{i=1}^n \qquad \{c_i\}_{i=1}^n \tag{36}$$

of $2n$ *non-zero* admissible numbers determined by the following $(2n - 3)$ parameters:

$$\{a_i\}_{i=1}^{n-1}, \qquad \{c_i\}_{i=1}^{n-2} \tag{37}$$

Since the numbers (37) are non-zero, one can choose two real numbers α, γ such that

$$0 < \alpha < a_i \quad (i = 1, 2, \ldots (n-1)) \tag{38}$$

$$0 < \gamma < c_i \quad (i = 1, 2, \ldots (n-2)) \tag{39}$$

Consider the following $[2(n + 1) - 3]$ numbers

$$\{a_i\}_{i=1}^{n-1}, \quad \alpha, \quad \{c_i\}_{i=1}^{n-2}, \quad \gamma \tag{40}$$

and consider them as possible parameters for a set of $2(n + 1)$ admissible numbers for the correlation $\mathrm{Corr}_p(A, B) > 0$; i.e. substitute the numbers in (40) into (28)–(30) (with the identifications $a_n = \alpha$, $c_{n-1} = \gamma$) to obtain c'_n, c'_{n+1}, a'_{n+1}. The $2(n + 1)$ numbers

$$\{a_i\}_{i=1}^{n-1}, \alpha, \alpha'_{n+1}; \qquad \{c_i\}_{i=1}^{n-2}, \gamma, c'_n, c'_{n+1} \tag{41}$$

so obtained will satisfy (21)–(23) but they are not necessarily admissible numbers because the numbers c'_n, c'_{n+1}, a'_{n+1} might not satisfy (24)–(25). However, for any n the formulas (28)–(30) considered as functions of the parameters $\{a_i\}_{i=1}^{n-1}$, $\{c_i\}_{i=1}^{n-2}$, are all continuous functions of these parameters; furthermore, it is clear from the formulas (28)–(30) that if α and γ jointly tend to zero:

$$(\alpha + \gamma) \to 0 \tag{42}$$

then

$$a'_{n+1} \to a_n, \quad c'_n \to c_{n-1}, \quad c'_{n+1} \to c_n \tag{43}$$

It follows then that for sufficiently small, non-zero α and γ the numbers c'_n, c'_{n+1}, a'_{n+1} will satisfy (24)–(25); consequently, for sufficiently small α and γ, the numbers (41) are $2(n+1)$ admissible numbers for the correlation $\mathrm{Corr}_p(A, B) > 0$.

Step 2 By Stone's representation theorem of Boolean algebras we may assume that Ω is a set of subsets of a set X. Let Ω_i and $X_i (i = 1 \ldots n)$ be n identical copies of Ω and X, respectively, and for all $i = 1 \ldots n$ let h_i denote the Boolean algebra isomorphisms between Ω and Ω_i. Let $X' = \cup X_i$. Let Ω' be the set of subsets in X' having the form $h_1(Y_1) \cup \cdots \cup h_n(Y_n)$, i.e. define

$$\Omega' = \{(h_1(Y_1) \cup \cdots \cup h_n(Y)) | Y_i \in \Omega(i = 1, \ldots n)\}$$

It is routine to verify that Ω' is a Boolean algebra of subsets of X' (with respect to the set theoretical operations) and that he map h defined by

$$h : \Omega \to \Omega', \quad X \mapsto (h_1(Y) \cup \cdots \cup h_n(Y))$$

is a Boolean algebra embedding of Ω into Ω'.

We now define a measure p' on Ω' in such a way that (Ω', p') becomes an extension of (Ω, p). Let $r_i^k (k = 1, 2, 3, 4; i = 1 \ldots n)$ be $4n$ real numbers in the interval $[0, 1]$ such that $\sum_{i=1}^n r_i^k = 1$ for all k. One can define a p' measure on Ω' by

$$p'(h_1(Y_1) \cup \cdots \cup h_n(Y_n)) \equiv \sum_{i=1}^{n} (r_i^1 p(Y_i AB) + r_i^2 p(Y_i A\overline{B} + r_i^3 p(Y_i \overline{A}B) + r_i^4 p(Y_i \overline{AB}))$$

Since AB, $A\overline{B}$, $\overline{A}B$ and $\overline{AB}$ are disjoint and their union is X it follows that

$$p'(h_1(Y) \cup \cdots \cup h_n(y)) = p'(h(Y)) = p(Y) \qquad Y \in \Omega$$

Hence (Ω', p') is indeed an extension of the original probability space (Ω, p).

Let $\{a_i\}_{i=1}^n, \{c_i\}_{i=1}^n$ be a set of admissible numbers and choose r_i^k as follows:

$$r_i^1 = \frac{c_i a_i^2}{p(AB)}$$

$$r_i^2 = r_i^3 = \frac{c_i a_i[1 - a_i]}{p(\overline{A}B)}$$

$$r_i^4 = \frac{c_i[1 - a_i)]^2}{p(\overline{AB})}$$

We claim that the following partition in Ω' is a Reichenbachian common cause system of size n for the correlation:

$$C_1 = h_1(X) \cup h_2(\emptyset) \cup \ldots \cup h_n(\emptyset)$$

$$C_2 = h_1(\emptyset) \cup h_2(X) \cup \ldots \cup h_n(\emptyset)$$
$$\vdots$$
$$C_i = h_1(\emptyset) \cup \ldots \cup h_i(X) \cup \ldots \cup h_n(\emptyset)$$
$$\vdots$$
$$C_n = h_1(\emptyset) \cup \ldots \cup h_i(\emptyset) \cup \ldots \cup h_n(X)$$

To see this one can check by explicit calculation that the followings hold:

$$p'(h_1(\emptyset) \cup \cdots \cup h_i(\Omega) \cup \cdots \cup h_2(\emptyset)) = p'(C_i) = c_i \tag{44}$$

$$p'(h_1(A) \cup \cdots \cup h_n(A) | h_1(\emptyset) \cup \cdots \cup h_i(\Omega) \cup \cdots \cup h_2(\emptyset)) = p'(A|C_i) = a_i \tag{45}$$

$$p'(h_1(B) \cup \cdots \cup h_n(B) | h_1(\emptyset) \cup \cdots \cup h_i(\Omega) \cup \cdots \cup h_2(\emptyset)) = p'(B|C_i) = a_i \tag{46}$$

Since the admissible numbers have been chosen precisely so that (12) and (13) are satisfied, $\{C_i\}_{i=1}^n$ above is indeed a conditionally identical common cause system of size n in $\{C_i\}_{i=1}^n$. Thus we have proven the theorem that any classical probabilistic measure space (Ω', p') can be extended in such a way that the extension contains a conditionally identical common cause systems of size n for the non-strict correlation.

5. CONCLUSIONS

The significance of Theorem 1 is straightforward. If for a pair of events (A, B) there exists a conditionally identical common cause system then the common cause will result in an exchangeable positive correlation. Thus conditionally identical common cause systems are apt tools for explaining exchangeable correlations.

Theorem 2 argues in the reverse mode. Exchangeable correlations can be explained by conditionally identical common cause systems by extending the original algebra of the correlation. Theorem 2 is a special case of a more general principle which is called Reichenbach's Common Cause Principle. This principle is the claim that correlating events which do not causally interact have a (Reichenbachian) common cause. In (Hofer-Szabó and Rédei, 1999) it was proven that Reichenbach's Common Cause Principle is true on the algebraic level in the sense that, given a correlation in a classical probability space (Ω, p), this probability space can be extended in such a way that the larger space contains a Reichenbachian common cause for the correlation. In (Hofer-Szabó and Rédei, 2006) this result has been generalized for any Reichenbachian common cause system in the sense that given a correlation in a classical probability space (Ω, p) and given any real number $n > 2$, (Ω, p) can be extended in such a way that the larger space contains a Reichenbachian common cause system of size

n for the correlation. Theorem 2 is a special case of this theorem since conditionally identical common cause systems are Reichenbachian common cause systems.

The real content of the Hypothesis of Conditionally Identical Probabilities lies in the extra assumption of Theorem 2, namely that exchangeability of the correlating pair derives from the conditional identity of the common cause. Or expressed in the language of symmetries, the permutation symmetry of the phenomenological level derives from the permutation symmetry of the hidden causal source. How legitime is this assumption? The answer depends on how much we expect from a causal explanation.

In the one hand, we might have good reasons to be more modest. Even the most simple common-causal explanation of the EPR experiment, namely the explanation via factorizing partitions imply Bell's theorems which contradicts quantum mechanical predictions. Since conditionally identical common cause systems are Reichenbachian common cause systems and Reichenbachian common cause systems are factorizing partitions, strengthening the requirements of the causal explanation is not the right way to account for quantum mechanical correlations.

In the other hand, concluding from spatial or other phenomenological symmetries of a system to symmetries of the hidden physical background is a natural type of reasoning in theoretical physics. The metaphysical reasoning in the Hypothesis of Conditionally Identical Probabilities has similar nature: one concludes from the symmetry of the *explanans* to the symmetry of the causal *explanandum*. Returning to spin experiment mentioned in the Introduction the axial symmetry of the whole setup together with the fact that the spin is a two-valued observable garantees invariance of the joint probability under permutation of the events which is just exchangeability. The Hypothesis of Conditionally Identical Probabilities then requires that the same type of invariance has to hold on the level of the common cause, which means that the common cause system has to be conditionally identical. If we strive for a full-fledged causal explanation then the causal *explanandum* has to account for all the relevant features of the *explanans*.

ACKNOWLEDGMENT

Work supported by János Bolyai Research Scholarship of the Hungarian Academy of Science.

REFERENCES

De Finetti, B. (1964). Foresight: Its logical laws, its subjective sources. In *Studies in Subjective Probabilitiy*, Kyburg, H. E. and Smokler, H. E., eds., Wiley, pp. 93–158.

Hofer-Szabó, G., Rédei, M., and Szabó, L. E. (1999). On Reichenbach's common cause principle and on Reichenbach's notion of common cause, *The British Journal for the Philosophy of Science* **50**, 377–399.

Hofer-Szabó, G., Rédei, M., and Szabó, L. E. (2002). Common causes are not common common causes, *Philosophy of Science* **69**, 623–633.

Hofer-Szabó, G. and Rédei, M. (2004). Reichenbachian common cause systems, *International Journal of Theoretical Physics* **34**, 1819–1826.

Hofer-Szabó, G. and Rédei, M. (2006). Reichenbachian common cause systems of arbitrary finite size exist, *Foundations of Physics* (forthcomming).

Jeffrey, R. (2004). *Subjective Probability*, Cambridge University Press.

Reichenbach, H. (1956). *The Direction of Time*, University of California Press, Berkeley.